LOYOLA COLLEGE (AUTONOMOUS), CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FIRST SEMESTER – **NOVEMBER 2015**

MT 1816 – REAL ANALYSIS

Date : 05/11/2015 Dept. No. Max. : 100 Marks

Time : 01:00-04:00

Answer all Questions. All questions carry equal marks.

1. (a) Give an example of the following, or state why no such example exists.

(i) A monotone function f: [0,1]→R which is not Riemann integrable.

(ii) Suppose that f(x) = $\begin{cases} 0 & 0 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}$ and $\propto = x^2$. Give an example of a partition P of [0, 1] such that $U(P, f, \propto) - L(P, f, \propto) < \frac{1}{4}$.

(OR)

(b) Define a refinement of a partition P. If P^* is a refinement of P then prove that $L(P, f, \propto) \le L(P^*, f, \propto)$ and $U(P^*, f, \propto) \le U(P, f, \propto)$. (5 marks)

(c) (i) If $f \in \Re(\alpha)$ on [a, b] and a <c <b, then prove that $f \in$ □(α) on [a, c] and on [c,b] and $\int_a^b f d \propto = \int_a^c f d \propto + \int_c^b f d \propto$

(ii) If $f \in \Re(\alpha)$ on [a, b] and if $|f(x)| \le M$ on [a, b], then prove that $\left|\int_a^b f d \propto\right| \le M[$□$(b) -$□$(a)]$.

(OR)

(d) (i) Assume α increases monotonically and $\alpha' \in$ □ on [a, b]. Let f be a bounded real function on [a, b]. Then prove that $f \in \Re(\alpha)$ if and only if $f\alpha'$ □ □. In that case $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.

(ii) State and prove the fundamental theorem of calculus. **(10+5marks)**

2. (a) Prove that for, $f_n(x) = \frac{\sin nx}{\sqrt{n}}, x\ real, n = 1,2 \ldots,$

$\lim_{n\to\infty} f_n'(0) \ne f'(0)$.

(OR)

(b) Suppose $\{f_n\}$ is a sequence of continuous functions on a set E and $|f_n(x)| \le M_n, x \in E, n = 1,2 \ldots$ then prove that $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

(5 marks)

(c) If $\{f_n\}$ is a sequence of differentiable functions on [a, b] such that $\{f_n(x_0)\}$ converges for $x_0 \in [a, b]$ and $\{f_n'\}$ converges uniformly on [a, b] then prove that $\{f_n\}$ converges uniformly on [a, b] to a function f and $\lim_{n\to\infty} f_n'(x) = f'(x)$.

(OR)

(d) State and prove the Stone-Weierstrass theorem **(15 marks)**

3. (a) If $f(x) \sim \sum_{n=0}^{\infty} c_n \emptyset_n(x)$, where $f \in L^2(I)$ and $\{\emptyset_0, \emptyset_1 \ldots\}$ be orthonormal on I, then prove that the series $\sum |c_n|^2$ converges and $\sum |c_n|^2 \leq \|f\|^2$ and $\sum |c_n|^2 = \|f\|^2$ if and only if $\lim_{n \to \infty} \|f - s_n\| = 0$

(OR)

(b) State and prove the Riesz-Fischer theorem. **(5 marks)**

(c) If $f \in L[0,2\pi]$, f is periodic with period 2π, then prove that the Fourier series generated by f converges for a given value of x if and only if for some $\delta < \pi$, $\lim_{n \to \infty} \frac{2}{\pi} \int_0^{\delta} \left(\frac{f(x+t)+f(x-t)}{2}\right) \frac{\sin\left(n+\frac{1}{2}\right)t}{t} dt$ exists and in this case this limit is the sum of the series. **(15 marks)**

(OR)

(d) (i) Define Dirichlet's kernel and prove that $\frac{1}{2} + \sum_{k=1}^{n} \cos kx = \frac{\sin(2n+1)\frac{x}{2}}{2\sin\frac{x}{2}}, x \neq 2m\pi$

(ii) If $f \in L[0,2\pi]$, f is periodic with period 2π and $\{s_n\}$ is a sequence of partial sums of Fourier series generated by f, $s_n = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx), n = 1,2 \ldots$ then prove that $s_n(x) = \frac{2}{\pi} \int_0^{\pi} \frac{f(x+t)+f(x-t)}{2} D_n(t) dt$ **(5+10 marks)**

4. (a) If A, B $\in$ L(R^n, R^m) and c is a scalar then prove the following:

(i) $\|A + B\| \leq \|A\| + \|B\|$

(ii) $\|cA\| = |c| \|A\|$

(iii) $\|A - C\| \leq \|A - B\| + \|B - C\|$, where A, B, C $\in$ L (R^n, R^m).

(OR)

(b) Prove that the set of all invertible linear operators on R^n, Ω is an open subset of $L(R^n)$ **(5 marks)**

(c) State and prove the inverse function theorem.

(OR)

(d) State and prove the implicit function theorem. **(15 marks)**

5. (a) Explain rectilinear coordinate system with algebraic and geometric approach.

(OR)

(b) Derive the derivative of x^n. **(5 marks)**

(c) Derive the D'Alembert's approach towards characterizing solution of 1-D wave equation.

(OR)

(d) Derive the expression for Newton's Law of Cooling.

(15 marks)

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